An Analysis of Chopper Circuits

Mikio NAKATSUYAMA

Department of Electrical Engineering, Faculty of Engineering

I. INTRODUCTION

Chopper circuits have been widely used as DC-AC convertor or AC-DC convertor of DC amplifier. It is necessary to compute the transfer function of chopper circuits, when DC amplifier is used as one of elements of operational amplifier or automatic control system. Although some papers[1)~4)] concerning transfer function of chopper circuits have already been published, it seems quite difficult to calculate the transfer function of complicated networks.

In this paper, the formula of the transfer function (which is easily calculated from the elements of F-matrix of circuits) will be derived from the use of the properties of voltage and current of chopper's terminal. The formula is valid, even when phases or duty ratios of modulating- or demodulating-chopper are different from each other and the optimum value of duty ratio will be given by this formula. The transfer function of the chopper modulating circuit containing a reasonant element is also given.

II. SOME PROPERTIES OF CHOPPER CIRCUITS

Chopper circuits may be represented as Fig. 1. Inserting network N_2 as

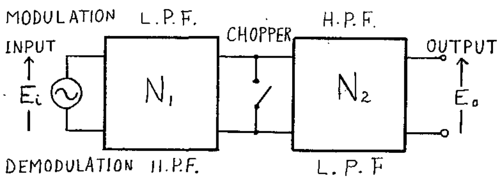


Fig.1 Typical chopper circuit.

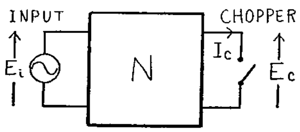


Fig.2 Chopper circuit represented as a two-port network.

one-port impedance into network N_1, we can show the more general representation of chopper circuits (Fig. 2). It is convenient to define the function $f(t)$ which is 1, when the terminal of chopper is open, and which is 0, when closed.

$$f(t) = \frac{1}{2}\left[A_o + 2\sum_{n=1}^{\infty}\left\{A_n \cos(n\omega_c t) + B_n \sin(n\omega_c t)\right\}\right] , \qquad (1)$$

$$\left.\begin{aligned} A_n &= \frac{2}{T}\int_{\varphi-\frac{l}{2}}^{\varphi+\frac{l}{2}} \cos(n\omega_c t)\,dt \\ B_n &= \frac{2}{T}\int_{\varphi-\frac{l}{2}}^{\varphi+\frac{l}{2}} \sin(n\omega_c t)\,dt \end{aligned}\right\}, \qquad (2)$$

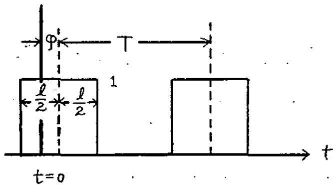


Fig.3 $f(t)$.

where ω_c is chopping frequency and $T = 2\pi/\omega_c$. Fig. 3 shows this function schematically. The more simple representation of Eq. (1) is

$$f(t) = \frac{1}{2}\left[a_o + 2\sum_{n=1}^{\infty} \dot{a}_a \cos(n\omega_c t+\phi_n)\right], \quad (3)$$

where $a_o = A_o$, $a_n = \sqrt{A_n^2+B_n^2}$ and $\phi_n = -\tan^{-1}(B_n/A_n)$, or this equation can be represented as

$$f(t) = \frac{1}{2} R_e\left[a_o + 2\sum_{n=1}^{\infty} \dot{a}_n e^{jn\omega ct}\right], \qquad (4)$$

where $\dot{a}_n = a_n e^{j\phi_n}$.

If $\varphi = o$,

$$f(t) = \frac{1}{2}\left[a_o + 2\sum_{n=1}^{\infty} a_n \cos(n\omega_c t)\right]. \qquad (5)$$

Clearly, the following equations hold at chopper's terminal.

$$f(t)\cdot E_c(t) = E_c(t) \qquad (6)$$

$$f(t)\cdot I_c(t) = O, \qquad (7)$$

where $E_c(t)$ and $I_c(t)$ are repsectively voltage and current at chopper's terminal. Fqs. (6) and (7) indicate important properties of chopper circuit.

The network shown in Fig. 1 and 2 contains low-pass and high-pass filter. For simplicity, assume at first that chopper circuits consist of only C and R. If the time constant of circuit element is C_iR_i, chopper circuit will in general satisfy following condition,

$$\left.\begin{aligned} &(n\omega_c \pm p)\, C_i\, R_i \gg 1 \qquad (n = 1, 2, 3, \cdots\cdots) \\ &\omega_c \gg p \end{aligned}\right\}, \qquad (8)$$

where p is input frequency. The network may consequently be regarded as one that consists of only resistance at higher frequencies, $(n\omega_c \pm p)$, where $n = +1, +2, \cdots\cdots$.

Then elements of F-matrix of network, A, B, C and D become

$$\left.\begin{aligned} A_n &= \bar{A}_n = A\,\infty, \quad B_n = \bar{B}_n = B\,\infty \\ C_n &= \bar{C}_n = C\,\infty, \quad D_n = \bar{D}_n = D\,\infty \end{aligned} \quad (n = \pm 1, \pm 2, \pm 3, \cdots\cdots)\right\}, \qquad (9)$$

where the subscript n indicates frequency of $(n\omega_c + p)$.

Ⅲ. CHOPPER-MODULATING CIRCUITS

At the steady state, chopper's terminal voltage $E_c(t)$ and current $I_c(t)$ may be represented as follows:

$$E_c(t) = R_e\dot{E}_c(t) = R_e\left[\sum_{n=0}^{\infty}\dot{E}_n e^{j(n\omega_c+p)t} + \sum_{n=-1}^{-\infty}\overline{\overline{E}_n\, e^{j(n\omega_c+pt)}}\right], \tag{10}$$

$$I_c(t) = R_e\dot{I}_c(t) = R_e\left[\sum_{n=0}^{\infty}\dot{I}_n e^{j(n\omega_c+p)t} + \sum_{n=-1}^{-\infty}\overline{\overline{I}_n\, e^{j(n\omega_c+p)t}}\right]. \tag{11}$$

From the definition of F-matrix,

$$\left.\begin{aligned} E_i &= \dot{A}_0\dot{E}_c + \dot{B}_0\dot{I}_0 \quad for \quad \omega = p, \\ O &= \dot{A}_n\dot{E}_n + \dot{B}_n\dot{I}_n \quad for \quad \omega = n\omega_c + p \quad (n = 1,\ 2,\ \cdots), \\ O &= \overline{A}_n\overline{E}_n + \overline{B}_n\overline{I}_n \quad for \quad \omega = n\omega_c + p \quad (n = -1,\ -2, \cdots). \end{aligned}\right\} \tag{12}$$

From Eqs. (11) and (12), $I_c(t)$ becomes

$$I_c(t) = R_e\left[\frac{\dot{E}_i}{\dot{B}_0}e^{jpt} - \sum_{n=0}^{\infty}\frac{\dot{A}_n}{\dot{B}_n}\dot{E}_n e^{j(n\omega_c+p)t} - \sum_{n=-1}^{-\infty}\overline{\frac{\dot{A}_n}{\dot{B}_n}\dot{E}_n e^{j(n\omega_c+p)t}}\right]. \tag{13}$$

Next, from Eq. (7), we may obtain

$$O = f(t)\cdot R_e\left[\frac{\dot{E}_i}{\dot{B}_0}e^{jpt} - \sum_{n=0}^{\infty}\frac{\dot{A}_n}{\dot{B}_n}\dot{E}_n e^{j(n\omega_c+p)t} - \sum_{n=-1}^{-\infty}\overline{\frac{\dot{A}_n}{\dot{B}_n}\dot{E}_n e^{j(n\omega_c+p)t}}\right]. \tag{14}$$

From Eq. (9), Eq. (14) becomes

$$f(t)\cdot Re\left(\frac{\dot{E}_i}{\dot{B}_0}e^{jpt}\right) = \frac{A\infty}{B\infty}\cdot R_e\,\dot{E}_c(t) + f(t)\cdot R_e\left[\left(\frac{\dot{A}_c}{\dot{B}_0} - \frac{A\infty}{B\infty}\right)\dot{E}_0 e^{jpt}\right]. \tag{15}$$

Compare the terms of Eq. (15) at each frequeny, then we can get voltage of each frequency at chopper's terminal represented as the function of E_i, or

$$E_0 = \frac{a_0}{(2-a_0)A\infty/B\infty + a_0\dot{A}_0/\dot{B}_0}\cdot\frac{\dot{E}_i}{\dot{B}_0} \quad for \quad \omega = p, \tag{16}$$

$$E_n = \frac{a_n}{(2-a_0)A\infty/B\infty + a_0\dot{A}_0/\dot{B}_0}\cdot\frac{\dot{E}_i}{B_0} \quad for\ \omega = n\omega_c + p\ (n=1,2,\cdots), \tag{17}$$

$$\overline{E}_n = \overline{\frac{a_n}{(2-a_0)A\infty/B\infty + a_0\dot{A}_0/\dot{B}_0}\cdot\frac{\dot{E}_i}{\dot{B}_0}} \quad for\ \omega = n\omega_c + p(n=-1,-2,\cdots). \tag{18}$$

The sum of Eqs. (16), (17) and (18), can be written as follows:

$$E_c(t) = f(t)\cdot R_e\left[\frac{2}{(2-a_0)A\infty/B\infty + a_0\dot{A}_0/\dot{B}_0}\cdot\frac{\dot{E}_i}{\dot{B}_0}e^{jpt}\right]. \tag{19}$$

If $Y_{oc}(j\omega)$ denotes the transfer function from chopper's terminal to output terminal and if $Y_{oi}(j\omega)$ the function from input terminal to output terminal,

$$|\,Y_{oc}[j(n\omega_c+p)]\,| = Y_{oc}(\infty),\ (n = \pm 1,\ \pm 2,\ \cdots). \tag{20}$$

Output voltage $E_o(t)$ is represented as

$$E_o(t) = R_e\left[Y_{oi}(jp)\,\dot{E}_i e^{jpt}\right] + R_e\left[Y_{oc}(jp)\frac{a_0}{(2-a_0)A\infty/B\infty + a_0\dot{A}_0/\dot{B}_0}\frac{\dot{E}_i}{\dot{B}_0}e^{jpt}\right]$$

$$+\,Y_{oc}(\infty)\cdot\left[f(t)-\frac{a_o}{2}\right]\cdot R_e\left[\frac{2}{(2-a_o)A\infty/B\infty+a_o\dot{A}_o/\dot{B}_o}\cdot\frac{\dot{E}_i}{\dot{B}_o}\,e^{jpt}\right]. \tag{21}$$

Therefore, the transient phenomena of chopper-modulating circuits can be investigated by taking the Laplace transform of Eq. (21). The product term of $f(t)$ may be thought the result of chopping in pure resistance network, and then $f(t)$ can be treated as a constant term. Considering this remark, the Laplace transform of Eq. (21) is given as the form of

$$E_o(s)=Y_{oi}(s)\cdot E_i(s)+Y_{oc}(s)\cdot\frac{a_o}{(2-a_o)A\infty/B\infty+a_oA_o(s)/B_o(s)}\cdot\frac{E_i(s)}{B_o(s)}$$

$$+\,Y_{oc}(\infty)\cdot\frac{2}{(2-a_o)A\infty/B\infty+a_oA_o(s)/B_o(s)}\cdot\frac{E_i(s)}{B_o(s)}\cdot\left[f(t)-\frac{a_o}{2}\right]\cdot \tag{22}$$

Ⅳ CHOPPER-DEMODULATING CIRCUIT

The output of the chopper-modulating circuit is applied as input to the chopper-demodulating circuit, so that the input is represented as follows:

$$E_i(t)=\left[f_1(t)-\frac{a_{10}}{2}\right]\cdot R_e\left[E_i\,e^{jpt}\right]. \tag{23}$$

When the component of fundamental frequency is applied as input to the chopper-demodulating circuit, the same formula as the chopper-modulating circuit holds.

If $f_2(t)$ denotes the function representing the performance of chopper of demodulating circuit, there holds in steady state the following relation, like Eq. (12),

$$\left.\begin{aligned}
&O=\dot{A}_o\dot{E}_o+\dot{B}_o\dot{I}_o \quad for \quad \omega=p,\\
&\frac{a_{1n}}{2}\ \ddot{E}_i=\dot{A}_n\dot{E}_n+\dot{B}_n\dot{I}_n \quad for \quad \omega=n\omega_c+p\ (n=1,2,3\cdots),\\
&\frac{a_{1n}}{2}\ \overline{E}_i=\overline{A}_n\overline{E}_n+\overline{B}_n\overline{I}_n \quad for \quad \omega=n\omega_c+p\ (n=-1,-2,-3),\cdots)\cdot
\end{aligned}\right\} \tag{24}$$

Then,

$$I_c(t)=R_e\Bigg[\sum_{n=1}^{\infty}\frac{a_{1n}}{2\ddot{B}_n}\dot{E}_i\,e^{j(n\omega c+p)t}+\overline{\sum_{n=-1}^{-\infty}\frac{a_{1n}}{2\ddot{B}_n}\cdot\dot{E}_i\,e^{j(n\omega c+p)t}}$$

$$-\sum_{n=0}^{\infty}\frac{\dot{A}_n}{\dot{B}_n}\dot{E}_n\,e^{j(n\omega c+p)t}-\overline{\sum_{n=-1}^{-\infty}\frac{\dot{A}_n}{\dot{B}_n}\dot{E}_n\,e^{j(n\omega c+p)t}}\Bigg]$$

$$=\frac{1}{B\infty}\left(f_1(t)-\frac{a_o}{2}\right)\cdot R_e\left[\dot{E}_{id}\,e^{jpt}\right]-\frac{A\infty}{B\infty}R_e\dot{E}_c(t)-R_e\left\{\left[\frac{\dot{A}_o}{\dot{B}_o}-\frac{A\infty}{B\infty}\right]\dot{E}_o\,e^{jpt}\right\} \tag{25}$$

Multiplying Eq. (25) by $f_2(t)$ and using Eqs. (6) and (7), we get

$$\frac{1}{B\infty}\left[f_3(t)-\frac{a_{10}}{2}f_2(t)\right]\cdot R_e\left[\dot{E}_{id}\,e^{jpt}\right]$$

$$=\frac{A\infty}{B\infty}R_eE_c(t)+f_2(t)R_e\left\{\left[\frac{\ddot{A}_o}{\dot{B}_o}-\frac{A\infty}{B\infty}\right]\cdot\dot{E}_o\,e^{jpt}\right\}, \tag{26}$$

where $f_3(t) = f_1(t) \cdot f_2(t)$. Compare the terms of Eq. (26) at each frequency, then we get voltages of each frequency represented as follows:

$$E_o = \frac{a_{3o} - a_{1o}\, a_{2o}/2}{(2-a_{2o})\, A\infty/B\infty + a_{2o}\dot{A}_o/B_o} \cdot \frac{E_i}{B\infty} \quad for\ \omega = p, \tag{27}$$

$$E_n = \left[\frac{a_{3n}B\infty}{2A\infty} - \frac{a_{2n}[a_1+a_{3o}(A_oA\infty/B_oB\infty-1)]}{2[(2-a_{2o})A\infty/B\infty+a_{2o}A_o/B_o]}\right]\frac{E_i}{B\infty} for\ \omega = n\omega_c+p(n=1,2,\cdots), \tag{28}$$

$$\overline{E}_n = \left[\overline{\frac{a_{3n}B\infty}{2A\infty} - \frac{a_{2n}[a_{1o}+a_{3o}(A_oA\infty/B_oB\infty-1)]}{2[(2-a_{2o})A\infty/B\infty+a_{2o}\dot{A}_o/\dot{B}_o} \frac{E_i}{B\infty}}\right] for\ \omega=\omega_c+p(n=-1,-2,)\cdots. \tag{29}$$

By summing Eqs. (27), (28) and (29), we get chopper's terminal voltage,

$$E_c(t) = f_3(t) \cdot R_e[(\dot{E}_{id}/A\infty)e^{jpt}]$$
$$-f_2(t) \cdot R_e\left\{\frac{a_{1o}+a_{3o}(A_oB\infty/B_oA\infty-1)}{(2-a_{2o})A\infty/B\infty+a_{2o}\dot{A}_o/B_o}\,\frac{E_i}{B\infty}\,e^{jpt}\right\}. \tag{30}$$

If $Y_{oc}(j\omega)$ denotes the transfer function from chopper's terminal to output terminal, output voltage is

$$E_{out} = Y_{oc}(jp)\,\frac{a_{3o} - a_{1o}a_{2o}/2}{(2-a_{2o})\,A\infty/B\infty + a_{2o}\,\dot{A}_o/\dot{B}_o} \cdot \frac{E_i}{B\infty}. \tag{31}$$

The components of higher frequencies may be sufficiently attenuated by R C low-pass filter and be regarded to be nearly zero; but ripple voltage at output terminal can be, if necessary, calculated from Eq. (30) as ω_c C_i R_i is not infinity. The Laplace transnsform of Eq. (31) is

$$E_{out}(s) = Y_{oc}(s)\,\frac{a_{3o} - a_{1o} \cdot a_{2o}/2}{(2-a_{2o})\,A\infty/B\infty + a_{2o}A_o(s)/B_o(s)}\;\frac{E_{id}(s)}{B\infty}. \tag{32}$$

Now, $f_2(t)$ does not suffer to any restrictions, so this formula is valid, even when duty ratios or phases of modulating-, or demodulating-chopper are different from each other.

V. WHOLE TRANSFER FUNCTION AND THE OPTIMUM VALUE OF DUTY RATIO

Combining Eq. (22) with Eq. (32), we obtain the whole transfer function

$$Y(s) = \left(Y_{doi}(s) + Y_{doc}(s)\,\frac{a_{2o}}{(2-a_{2o})B_d(s)\cdot A_d(\infty)/B_d(\infty) + a_{2o}A_d(s)}\right)$$
$$\cdot \left(Y_{moi}(s)+Y_{moc}(s)\,\frac{a_{1o}}{(2-a_{1o})B_m(s)A_m(\infty)/B_m(\infty)+a_{1o}A_m(s)}\right)$$
$$+ Y_{doc}(s) \cdot \frac{a_{3o} - a_{1o}a_{2o}/2}{(2-a_{2o})A_d(\infty) + a_{2o}B_d(\infty)A_d(s)/B_d(s)}$$
$$\cdot\, Y_{moc}(s) \cdot \frac{2}{(2-a_{1o})B_m(s)A_m(\infty)/B_m(\infty)+a_{1o}A_m(s)}, \tag{33}$$

where subscript m denotes the element of modulating circuit and subscript d denotes that of demodulating circuit. The first term of Eq. (33) indicates the effect of the component of fundamental frequency and is, generally, rather small. The second term indicates the effect of the components of the higher frequency.

The duty ratio is the ratio of the contact time of chopper to switching period, so it corresponds to $a_o/2$. If the first term of Eq. (33) is zero, the optimum value of duty ratio will be easily calculated. When input is DC, the second term takes the real and largest value, and is represented as follows:

$$Y(o) = Y_{doc}(o)\cdot Y_{moc}(o)\ \frac{a_{3o}-a_{1o}a_{2o}/2}{(2-a_{2o})A_d(\infty)+a_{2o}B_d(\infty)A_d(o)/B_d(o)} \cdot \frac{2}{(2-a_{1o})B_m(o)A_m(\infty)/B_m(\infty)+a_{1o}A_m(o)}\ . \tag{34}$$

Therefore. the proplem is to get the value of a_o that makes Y(o) largest. When $f_1(t) = f_2(t)$, $a_{1o} = a_{3o} = a_o$. If $(2-a_o)/a_o = x$, by differerntiating Eq. (34) with x, the optimum value of a_o is

$$a_o = \frac{2}{1+\left(\frac{A_m(o)}{A_m(\infty)}\frac{B_m(\infty)}{B_m(o)}\frac{A_d(o)}{A_d(\infty)}\frac{B_d(\infty)}{B_d(o)}\right)^{\frac{1}{2}}}\ . \tag{35}$$

When $f_2(t) = 1-f_1(t)$, $a_{3o} = 0$ and $a_{2o} = (2-a_{1o})$. If $(2-a_{1o})/a_{1o} = x$, by differentiating Eq. (34) with x, we will get the optimum value of a_{10},

$$a_{10} = \frac{2}{1+\left(\frac{A_m(o)}{A_m(\infty)}\frac{B_m(\infty)}{B_m(o)}\frac{A_d(\infty)}{A_d(o)}\frac{B_d(o)}{B_d(\infty)}\right)^{\frac{1}{2}}}\ . \tag{36}$$

If $f_2(t)$ is any other function, it is diffcult to calculate the optimum value of a_o.

Ⅵ. BANDWIDTH LIMITATION

The frequency bandwidth applicable to Eq. (21) is rather wide, because p will be able to approach very closely to ω_c, only if the condition, $(\omega_c\text{-p})C_i R_i \gg 1$, is fulfiled. Moreover, the condition, $\omega_c \gg$ P, will be not necessary, if $\omega_c >$ p.

The frequency bandwidth applicable to Eq. (30) is the same as that of chopper-modulating circuit. It must, however, reproduce the input signal in chopper-modulating circuit, so bandwidth limitation occurs in connection with this.

The components of frequency p, $(\omega_c-\text{p})$, $(\omega_c+\text{p})$ and etc., appeare at chopper's terminal. Chopper-demodulating circuit must contain an ideal low-pass filter whose cutt-off frequency is $\omega_c/2$ to separate the component of frequency p from that of frquency $(\omega_c-\text{p})$; neverthless, the waveform distortion will occur if p $>\omega_c/2$. Chopper-demodulating circuit will, therefore, reproduce precicely the input signal, of which frequency bandwidth is $0 \leqq$ p $< \omega_c/2$, if an ideal low-pass filter exists. In fact, CR low-pass filter being used, in stead of an ideal low-pass filter, the freqnency bandwidth will be more narrow than the above-mentioned value.

VII. EXAMPLES

(a) Chopper-Modulating Circuit

Chopper-modulating circuit shown in Fig. 4 (a) is rewritten as Fig. 4

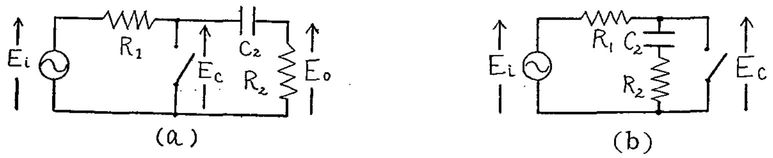


Fig. 4 Chopper-modulating circuit.

(b). The elements of F-matrix, A (s) and B (s), and the voltage transfer function, $Y_{oc}(s)$, of Fig. 4 (b) are

$$A(s) = 1 + \frac{SC_2R_1}{1+SC_2R_2}, \qquad B(s) = R_1,$$

$$Y_{oc}(s) = \frac{SC_2R_2}{1+SC_2R_2},$$

and

$$A(\infty) = 1 + \frac{R_1}{R_2}, \quad B(\infty) = R_1, \quad Y_{oc}(\infty) = 1 .$$

From Eq. (22), the transfer function is

$$Y(s) = \frac{a_o R_2^2 C_2 S}{a_o R_2 + (2-a_o)(R_1+R_2) + 2R_2C_2(R_1+R_2)S} + \frac{2R_2(1+SC_2R_2)}{a_oR_2+(2-a_o)(R_1+R_2)+2R_2C_2(R_1+R_2)S}\left[f(t) - \frac{a_o}{2}\right]. \quad (37)$$

The inverse transform of Eq. (37), when input is step function, that is, $E_i(s) = E_i/S$, will be represented as follows:

$$E_o(t) = \frac{a_oR_2}{2(R_1+R_2)}E_i\, e^{-\frac{t}{T_m}} + \frac{2R_2E_i}{2(R_1+R_2)-a_oR_1}\left[1-\frac{a_oR_1}{2(R_1+R_2)}e^{-\frac{t}{T_m}}\right]\cdot\left[f(t)=\frac{a_o}{2}\right],$$

where $T_m = \frac{2R_2C_2(R_1+R_2)}{a_oR_2+(2-a_o)(R_1+R_2)}$. When $R_1 = 1.01M\Omega$, $R_2 = 1.06M\Omega$, $C_2 = 0.0961\ \mu F$ and $a_o = 0.877$, Eq. (37) becomes

$$E_o(t) = E_i\left[0.224\, e^{-7.73t} + 0.650(1-0.214e^{-7.73t})\begin{Bmatrix} +\ 0.561 \\ or \\ -\ 0.439 \end{Bmatrix}\right].$$

The envelope of the transient response taken with the electro-magnetic oscillograph (shown in Fig. 5) is compared with the calculated one in Fig. 6.

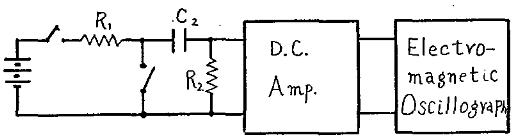


Fig.5 Chopper-modulating circuit and the block diagram measuring it's transient phenomena.

The experimental value at the vicinity of $t = O$ is slightly differnt from the calculated one, because of bandwidth limitation. Eq. (37) coincides with the result of reference 1.

(b) Chopper-Demodulating Circuit

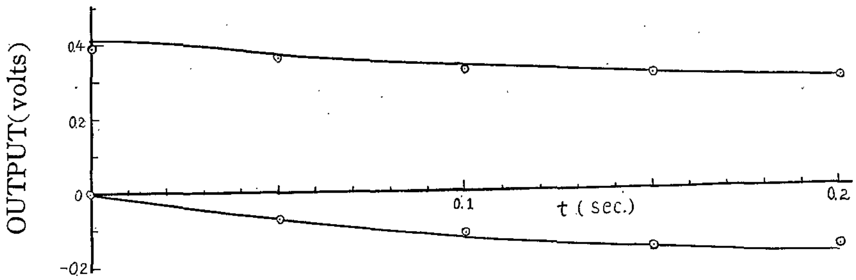
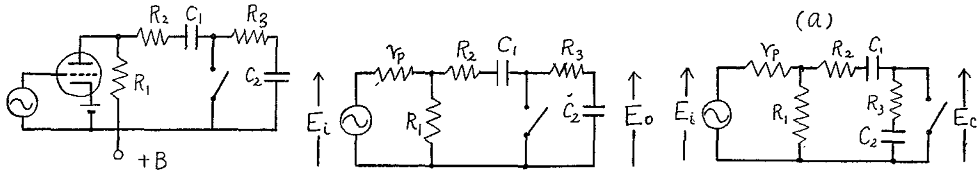


Fig .6 Transient response (envelope) of chopper-modulating circuit; full line, calculated; ◉, experimental. $E_i = 0.8$ volt. $\frac{\omega_c}{2\pi} = 100$(c/s).

The elements of F-matrix and voltage transfer function of chopper-demodulating circuit (Fig. 7 (c)) are



(a) (b) (c)

Fig. 7 Chopper-demodulating circuit.

$$A(s) = \frac{1}{1+SR_3C_2}\left[(1+r_p/R_1)(1+C_2/C_1)+SC_2\left\{(1+r_p/R_1)(R_2+R_3)+r_p\right\}\right]$$

$$\equiv \frac{1}{1+SR_3C_2}\,[a+bS],$$

$$B(s) = \frac{(1+r_p/R_1)+S[r_p+R_2+R_2r_p/R_1]C_1}{SC_1} \equiv \frac{c+dS}{SC_1},$$

$$Y_{co}(s) = (1+SC_2R_3)^{-1}, \quad A(\infty) = b(R_3C_2)^{-1},$$

$$B(\infty) = d/C_1, \qquad Y(\infty) = o.$$

From Eq. (32), the transfer function is

$$Y(s) = 2^{-1}(2-a_o)a_oR_2C_2(c+dS)(\alpha S^2+\beta S+\gamma)^{-1}, \tag{38}$$

where $\alpha = 2bdR_3C_2$, $\beta = (2-a_o)b(cR_3C_2+d)+a_oadR_3C_2$ *and* $\gamma = (2-a_o)bc$.

When output of chopper-modulating circuit is applied as the type of step function, the transient response will become, by using Eq (38),

$$E_o(t) = \frac{a_o(2-a_o)R_3C_2}{2\gamma}\left[c+\frac{\alpha}{\sqrt{\beta^2-\alpha\gamma}}\left\{S_2+\frac{d\gamma}{\alpha}\right)e^{S_1t}\right.$$

$$\left.-\left(S_1+\frac{d\gamma}{\alpha}\right)e^{S_2t}\right\}\Bigg], \tag{39}$$

where $S_1,\ S_2 = \dfrac{-\beta\pm\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}$.

(c) The Whole Transfer Function

With consideration of $Y_{doi} = Y_{moi} = O$ from Fig. 8 and Eqs. (34), (37) and (38), the whole transfer function, Y(s), will be written as follows:

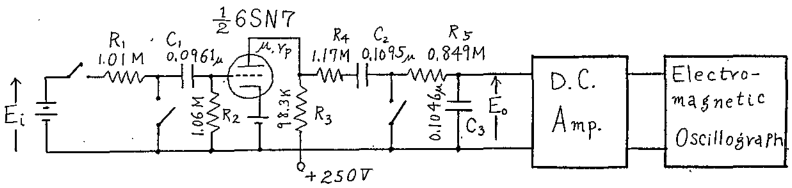


Fig. 8 An example of chopper amplifier and the block diagram measuring it's transient phenomena.

$$Y(s) = \frac{\mu a_0^2 dR_2R_5C_3C_1}{2\alpha(R_1+R_2)C_1}\cdot\frac{S^2}{(S-S_1)(S-S_2)(S-S_3)}$$

$$+\frac{\mu(2-a_o)aR_5C_3}{2\alpha(R_1+R_2)C_1}\cdot\frac{(c+dS)(1+SC_1R_2)}{(S-S_1)(S-S_2)(S-S_3)}, \tag{40}$$

where $S_1,\ S_2 = \dfrac{-\beta\pm\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}$ and $s_3 = -\dfrac{a_oR_2+(2-a_o)(R_1+R_2)}{2R_2C_1(R_1+R_2)}$.

The transient response to the step function input is

$$E_o(t) = k_o\,E_i\left\{-\frac{k_3}{S_1S_2S_3}+\frac{k_1S_1^2+k_2S_1+k_3}{S_1(S_1-S_2)(S_1-S_3)}\,e^{S_1t}\right.$$

$$\left.+\frac{k_1S_2^2+k_2S_2+k_3}{S_2(S_2-S_1)(S_2-S_3)}\,e^{S_2t}+\frac{k_1S_3^2+k_2S_3+k_3}{S_3(S_3-S_1)(S_3-S_2)}\,e^{s_3t}\right\}, \tag{41}$$

where $k_o \equiv \dfrac{\mu a_oR_5C_3}{2\alpha(R_1+R_2)c}$, $k_1 \equiv 2\,dC_1R_2$, $k_2 \equiv (2-a_o)d + C_1R_2c$ and $k_3 \equiv (2-a_o)\,c$.

Using the numerical value shown in Fig. 8, we get, from Eq. (41),

$$E_o(t) = E_i[2.18-7.82\,e^{-5.27t}-13.78\,e^{-9.29t}+19.43\,e^{-7.72t}],$$

where $\mu = 21.8$, $a_o = 0.877$ and $r_p = 13.9\,k\Omega$.

The experimental curve is compared with the calculated one in Fig. 9 .

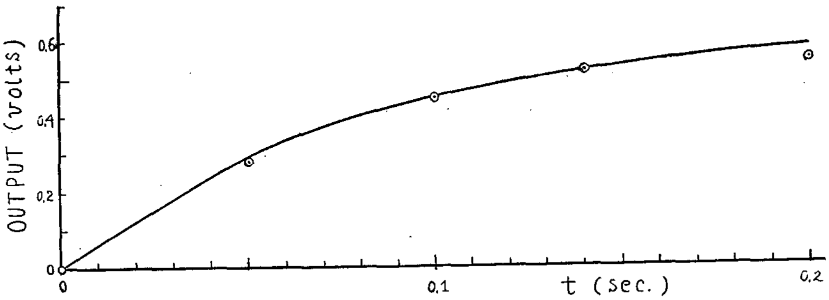


Fig. 9 Transient response of chopper amplifier to a step function input; full line, calculated; ◉, experimental;

$E_i = -0.3$ volt. $\frac{\omega_c}{2\pi} = 100$(c/s).

Ⅷ. THE CONSTANT CURRENT SOURCE

Hithereto, all input sources have been dealt as constant voltage sources, but input source may be sometimes regarded as constant current source. Then the transfer function of such a case may be easily calculated from equation,

$$\dot{I}_1 = \dot{C}\dot{E}_2 + \dot{D}\dot{I}_2 ,$$

by the same method as that used in constant voltage source. As for the result only, the transfer function of network containing a constant current source as input will be obtained by replacing A and B with C and D respectively.

Ⅸ. L C R CIRCUIT

(a) Non-Resonant Circuit

The transformer is sometimes used to get a larger gain. The simple equivalent circuit of a transformer shown in Fig. 10 (a) becomes the one shown in Fig. 10 (b) by neglecting L_2 that is generally small. The network containing L_1 will be regarded as a pure resistance network at the high frequencies. When the condition, $(n\omega_c \pm p)\ L \gg R$ (n = 1, 2, ……), is fulfiled, quite the same formula as that of RC network holds. When a servo-motor follows chopper-modulating circuit, the same formula is valid, if L of servo-motor has very high impedance at

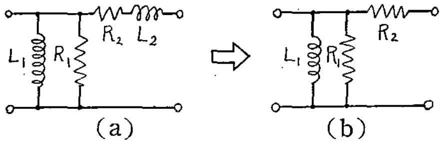
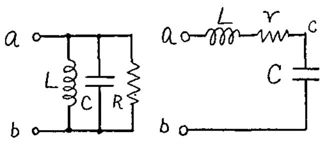


Fig. 10 A simple equivalent circuit of a transformer.

higher frequencies and if the terminal of L is regarded as open.

(b) Resonant Circuit

The resonant circuit consisting of L of the transformer and C will give larger gain to chopper-modulating circuit. Namely, the transfer function of chopper-modulating circuit containing L and C resonant circuit, will be given in case that Q of a resonant element is sufficiently high. The impedance of parallel resonant circuit (Fig. 11 (a)) is



(a) (b)

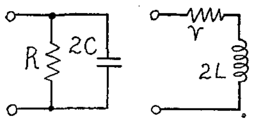
Fig. 11 (a) parallel-and (b)series-resonant circuit.

$$Z_p = \frac{1}{1/R+1/j\omega L+j\omega C} = \frac{1}{1/R+jC(\omega^2-\omega_o^2)/\omega} \doteqdot \frac{1}{1/R+j2pC}, \tag{42}$$

where $\omega_o^2 = (LC)^{-1}$, $\omega = \omega_o + p$ *or* $\omega = -\omega_o + p$ and the impedance of series resonant circuit (Fig. 11 (b)) is

$$Z_s = r + j\omega L + \frac{1}{j\omega C} = r + j\frac{L}{\omega}(\omega^2-\omega_o^2) \doteqdot r+j2LP, \tag{43}$$

where $\omega_o^2 = (LC)^{-1}$, $\omega = \omega_o+p$ *or* $\omega = -\omega_o+p$.



(a) (b)

Fig. 12 Equivalent circuits of (a) parallel- and (b) series-resonant circuit.

It is evident from Eqs. (42) and (43) that both Z_p and Z_s are the functions of p and the equivalent circuits may be represented as Fig. 12 (a) and (b). From these facts, Eq. (12) will be rewritten as follows:

$$\left.\begin{aligned} &E_i = \dot{A}_o\dot{E}_o + \dot{B}_o\dot{I}_o && \text{for } \omega = p, \\ &O = \dot{A}_1\dot{E}_1 + \dot{B}_1\dot{I}_1 && \text{for } \omega = \omega_c + p, \\ &O = \overline{A}_{-1}\overline{E}_{-1} + \overline{B}_{1-}\overline{I}_{-1} && \text{for } \omega = -\omega_c + p, \\ &O = A\infty E_n + B\infty I_n && \text{for } \omega = n\omega_c + p \ (n = \pm 2, \pm 3, \cdots). \end{aligned}\right\} \tag{12'}$$

The transfer function of the circuit containing a resonant element may be derivd by the almost same method as in the section Ⅲ, so we give only the result. If E_i, E_1 and $\bar{E}_{-1}$ denote the voltages of fundamental component, (ω_c+p) component, and (ω_c-p) component, respectively, we shall have the following relations:

$$\frac{E_i}{B_o}\begin{pmatrix} a_0 \\ a_1 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_0[o] & a_1(-1) & a_1(+1) \\ a_1(o) & a_o[-1] & a_2(+1) \\ a_1(o) & a_2(-1) & a_o[+1] \end{pmatrix} \cdot \begin{pmatrix} E_o \\ E_{-1} \\ E_{+1} \end{pmatrix}, \tag{44}$$

where $[i] = [\dot{A}_i/\dot{B}_i + \{(2-a_o)/a_o\}\{A\infty/B\infty\}]$ and $(i) = (\dot{A}_i/\dot{B}_i - A\infty/B\infty)$.

Setting $\Delta = a_o[o][a_o^2[-1][+1] - a_2^2(-1)(+1)] - a_1(o)[a_oa_1(-1)[+1] - a_1a_2(-1)$

$$(+1)]+a_1(o)[a_1a_2(-1)(+1)-a_oa_1[-1](+1)], \tag{45}$$

$$\Delta_o = \dot{B}_o^{-1}\{a_o[a_o^2[-1][+1]-a_2^2(-1)(+1)]-a_1[a_oa_1(-1)[+1]-a_1a_2(-1)(+1)]+a_1[a_1a_2(-1)(+1)-a_oa_1[-1](+1)]\}, \tag{46}$$

$$\Delta_{-1} = \dot{B}_o^{-1}\{-a_o[a_oa_1(o)[+1]-a_1a_2(o)(+1)]+a_1[a_o^2[o][+1]-a_1^2(o)(+1)]-a_1[a_oa_2[o](+1)-a_1^2(o)(+1)]\}, \tag{47}$$

$$\Delta_{+1} = \dot{B}_o^{-1}\{a_o[a_1a_2(o)(-1)-a_oa_1(o)[-1]]-a_1[a_oa_2[o](-1)-a_1^2(o)(-1)]+a_1[a_o^2[o](-1)-a_1^2(o)(-1)\}, \tag{48}$$

we shall obtain next solutions,

$$\left.\begin{aligned} E_o &= (\Delta_o/\Delta)\ E_i \\ E_{+1} &= (\Delta_+/\Delta)\ E_i \\ E_{-1} &= (\Delta_{-1}/\Delta)\ E_i \end{aligned}\right\}. \tag{49}$$

The relations $[+1] = [-1]$, and $(+1) = (-1)$, are naturally evident, and taking Eqs. (42) and (43) into account,

$$\Delta_{-1} = \Delta_{+1} = \Delta'(p) \tag{50}$$

holds; therefore,

$$E_{+1} = E_{-1} \tag{51}$$

and both voltages become functions of p. If Y_{pc} denotes the voltage transfer function from chopper's terminal to a parallel resonant element and if Y_{sc} denotes the current transfer function from chopper's terminal to a series resonant element, both Y_{pc} and Y_{sc} are functions of p and satisfy the following equations:

$$Y_{pc}(\omega_c+p) = Y_{pc}(-\omega_c+p) \equiv Y'_{pc}(p), \tag{52}$$

$$Y_{sc}(\omega_c+p) = Y_{sc}(-\omega_c+p) = Y'_{sc}(p). \tag{53}$$

If I_{+1} and I_{-1} denote currents at the frequency (ω_c+p) and $(-\omega_c+p)$ respectively, the voltage between the terminals a and c as shown Fig. 11 (b) are

$$\dot{E}_{L+1} = \dot{I}_{+1}\cdot j(\omega_c+p)L \doteqdot j\omega_c L I_{+1} \tag{54}$$

$$\dot{E}_{L-1} = \dot{I}_{-1}\cdot j(-\omega_c+p)L \doteqdot -j\omega_c L I_{-1} \tag{55}$$

i) Parallel Resonant Circuit

The voltages at the terminal of a parallel resonant element are

$$\dot{E}_{o+1} = Y_{pc}(\omega_c+p)\cdot(\Delta_{+1}/\Delta)\cdot\dot{E}_i = Y'_{pc}(p)\cdot(\Delta'(p)/\Delta)\cdot\dot{E}_i \tag{56}$$

$$\dot{E}_{o-1} = Y_{pc}(-\omega_c+p)\cdot(\Delta_{-1}/\Delta)\cdot\dot{E}_i = Y'_{pc}(p)\cdot(\Delta'(p)/\Delta)\cdot\dot{E}_i \tag{57}$$

Combine Eqs. (56) and (57), and we have both output voltage $E_{out}(t)$,

$$\begin{aligned} E_{out}(t) &= R_e\{\dot{E}_{o+1}\,e^{j(\omega_c+p)t}+\overline{E_{o-1}\,e^{j(-\omega cp)}}\} \\ &= R_e\{Y'_{pc}(p)\cdot(\Delta'(p)/\Delta)\cdot\dot{E}_i\,e^{j(\omega c+p)t}+\overline{Y'_{pc}(p)\cdot(\Delta'(p)/\Delta)\cdot\dot{E}_i e^{j(-\omega c+p)t}}\} \\ &= R_e\{2\,Y'_{pc}(p)\cdot(\Delta'(p)/\Delta)\cdot\dot{E}_i\,e^{jpt}\}\cos(\omega_c t), \end{aligned} \tag{58}$$

and the transfer function of the envelope,

$$Y_{pe} = 2\,Y'_{pc}(s)\cdot\frac{\Delta'}{\Delta}(s). \tag{59}$$

ii) Series Resonant Circuit

The voltage at L of a series resonant element is

$$E_{L+1} = j\omega_c L \cdot Y_{sc}(\omega_c + p) \cdot (\Delta_{+1}/\Delta) \cdot \dot{E}_i = j\omega_c L \cdot Y'_{sc}(p) \cdot (\Delta'/\Delta)(p) \cdot \dot{E}_i \qquad (60)$$

$$E_{L-1} = -j\omega_c L \cdot Y_{sc}(-\omega_c + p) \cdot (\Delta_{-1}/\Delta) \cdot \dot{E}_i = -j\omega_c L \cdot Y'_{sc}(p) \cdot (\Delta'/\Delta)(p) \cdot \dot{E}_i \cdot \qquad (61)$$

Consequently, the combined output voltage is

$$E_{out}(t) = R_e\{j\omega_c L \cdot Y'_{sc}(p) \cdot (\Delta'/\Delta)(p) \cdot \dot{E}_i \cdot e^{j(\omega_c + p)t}$$
$$\overline{-j\omega_c L \cdot Y'_{sc}(p) \cdot (\Delta'/\Delta)(p) \cdot \dot{E}_i \cdot e^{j(-\omega_c + p)t}}\}$$
$$= -R_e\{2\omega_c L \cdot Y'_{sc}(p) \cdot (\Delta'/\Delta)(p) \cdot \dot{E}_i \cdot e^{jpt}\} \ sin \ (\omega_c t). \qquad (62)$$

Therefore the transfer function of the envelope becomes

$$Y_{se}(s) = 2\omega_c L \cdot Y'_{sc}(s) \cdot (\Delta'/\Delta)(s). \qquad (63)$$

Eq. (49) is the complicated formula as it is, but it will take the simpler, following representation, if duty ratio is 1/2 and if $(o) = o$ in case where the network is rather simple:

$$\Delta = [o] \cdot [-1] \cdot [+1],$$

$$\Delta_{-1} = \frac{1}{\dot{B}_o} \frac{2}{\pi} [o] \cdot [+1],$$

$$\Delta_{+1} = \frac{1}{\dot{B}_o} \frac{2}{\pi} [o] \cdot [-1],$$

$$\therefore \quad \dot{E}_{-1} = \dot{E}_{+1} = \frac{\frac{2}{\pi}}{[+1]} \cdot \frac{\dot{E}_i}{\dot{B}_o}. \qquad (64)$$

iii) Example

Elements of F-matrix of Fig. 13 (c) are

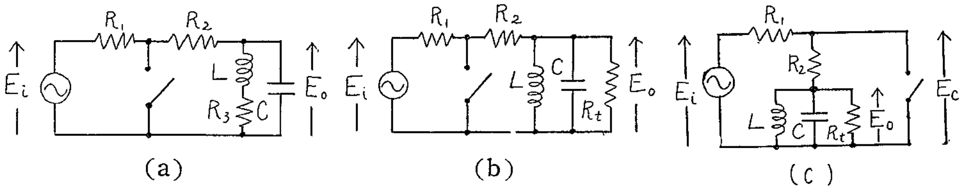


Fig .13 Chopper-modulating circuit containing a parallel resonant circuit. $R_t = L/R_3C$.

$$A = 1 + \frac{R_1}{R_2 + \dfrac{1}{1/R_t + 1/j\omega L + j\omega C}}, \qquad B = R_1,$$

$$A(o) = A(\infty) = 1 + R_1/R_2, \quad B(\infty) = R_1,$$

and if $a_o = 1$, $a_1 = 2/\pi$ and $a_2 = 0$,

$$A_{+1} = A_{-1} = 1 + \frac{R_1}{R_2 + \dfrac{1}{1/R_t + j2pc}} = 1 + \frac{R_1(1/R_t + j2pC)}{1 + R_2/R_t + j2pCR_2},$$

$$[-1]=[+1]=\frac{1}{R_1}+\frac{1/R_t+j2pC}{1+R_2/R_t+j2pCR_2}+\frac{1}{R_1}+\frac{1}{R_2}=\frac{2}{R_1}+\frac{1}{R_2}+\frac{1/R_t+j2pC}{1+R_2/R_t+j2pCR_2},$$

$$Y_{-1}=Y_{+1}=\frac{\dfrac{1}{1/j\omega L+j\omega C+1/R_t}}{R_2+\dfrac{1}{1/j\omega L+j\omega C+1/R_t}}=\frac{1}{1+R_2/R_t+j2pCR_2}\ .$$

Therefore,

$$\dot{E}_{-1}=\dot{E}_{+1}=\frac{(2/\pi)\ \dot{E}_i}{1/R_1+1/R_2+\dfrac{1/R_t+j2pC}{1R+R_2/R_t+j2pCR_2}}\cdot\frac{1}{R_1}$$

$$=\frac{(2/\pi)\dot{E}_i(1+R_2/R_t+j2pCR_2)}{(2+R_1/R_2)(1+R_2/R_t)+1/R_t+j2pC(2+2R_2/R_1)}\ \frac{1}{R_1}\ .$$

Output voltage is

$$E_{out}(t)=R_e\left\{\frac{(4/\pi)\ \dot{E}_i\ e^{jpt}}{(2+R_1/R_2)(1+R_2/R_t)+R_1/R_t+j4pC(R_1+R_2)}\right\}\ cos\ (\omega_c t)\ .$$

Settng $R_1=503\,k\Omega$, $R_2=508\,k\Omega$, $R_3=576\,\Omega$, $C=0.0182\,\mu F$ and $L=5.07H$, we obtain the transfer function of envelope, or

$$Y(jf)=\frac{6.46}{36.4+j2.29\,f}\ ,$$

and the theoretical curve is shown together with the experimental one in Fig. 14.

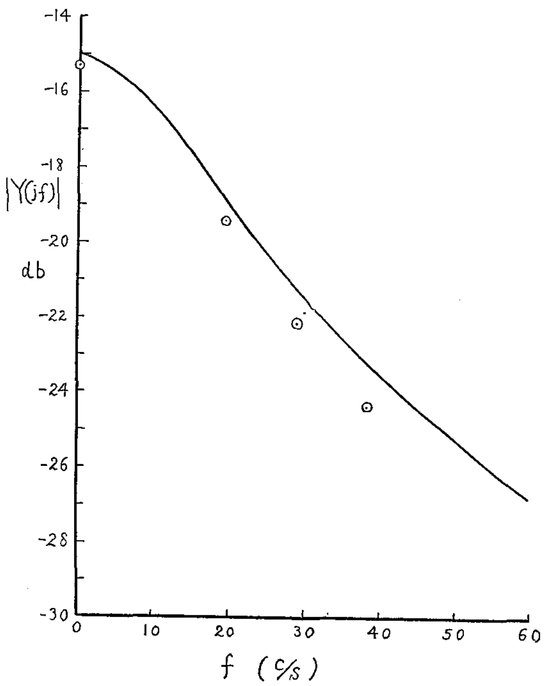


Fig. 14 Frequency-response characteristics of chopper-modulating circuit containing a parallel resonant circuit; full line, theoretical; ◉, experimental.

X. DEMODULATION OF BALANCED MOULATED WAVE

In case where modulating circuit is a static-type DC-AC convertor, or contains resonant element, output of modulating circuit becomes balanced modulated wave; that is, both the component of fundamental frequency and the carrier wave will not be transmitted, of course. Output of demodulating circuit will be easily calculated with the fundamental theory of this paper, when such balanced modulated wave is applied to demodulating circuit. The author gives the outline of deduction of the transfer function as follows.

Balanced modulated wave is represnted as

$$E_i(t)=E_i \cos(pt+\varphi)\cdot\cos(\omega_c t+\theta), \tag{65}$$

which yields the following complex representation,

$$\begin{aligned}\dot{E}_i(t) &= \frac{E_i}{2} e^{j\varphi} e^{j\theta} e^{j(\omega_c+p)t} + \frac{E_i}{2} e^{-j\varphi} e^{j\theta} e^{j(\omega_c-p)t}\\ &= \frac{1}{2}\left[\dot{E}_i e^{j(\omega_c+p)} e^{j\theta} + \overline{\dot{E}_i e^{j(-\omega_c+p)t} e^{-j\theta}}\right],\end{aligned} \tag{66}$$

where $\dot{E}_i = E_i e^{j\varphi}$. In steady state, F-matrix representaion is

$$\left.\begin{aligned} O &= \dot{A}_o\dot{E}_o + \dot{B}_o\dot{I}_o && \text{for } \omega=p,\\ \frac{1}{2}\overline{E}_i\overline{e^{-j\theta}} &= A\infty\ \overline{E}_{-1} + B\infty\ \overline{I}_{-1} && \text{for } \omega=\omega_c+p,\\ \frac{1}{2}E_i e^{j\theta} &= A\infty\ \dot{E}_{+1} + B\infty\ \dot{I}_{+1} && \text{for } \omega=\omega_c+p,\\ O &= A\infty\ E_n + B\infty\ I_n && \text{for } \omega=n\omega_c+p (n=\pm2,\pm3,\cdots).\end{aligned}\right\} \tag{67}$$

If $f(t)$ indicates the performance of demodulating chopper, it holds the relation as follows:

$$\begin{aligned}&f(t)\cdot R_e\left[\frac{1}{2B\infty}\dot{E}_i e^{-j\theta} e^{j(-\omega_c+p)t} + \frac{1}{2B\infty}\dot{E}_i e^{j\theta} e^{j(\omega_c+p)t}\right]\\ &= \frac{A\infty}{B\infty} R_e\,\dot{E}_c(t) + f(t)\cdot R_e\left[\left(\frac{\dot{A}_o}{\dot{B}_o}-\frac{A\infty}{B\infty}\right)\dot{E}_o e^{jpt}\right].\end{aligned} \tag{68}$$

Compare the coefficients of fundamental frequency, then we get

$$\frac{a_1\dot{E}_i}{4B\infty} e^{-j\theta} + \frac{a_1\dot{E}_i}{4B\infty} e^{j\theta} = \frac{A\infty}{B\infty}\dot{E}_o + \frac{a_o}{2}\left(\frac{\dot{A}_o}{\dot{B}_o}-\frac{A\infty}{B\infty}\right)\dot{E}_o, \tag{69}$$

which yields

$$\dot{E}_o = a_1 B_\infty^{-1}\left[(2-a_o)\frac{A\infty}{B\infty} + a_o\frac{\dot{A}_o}{\dot{B}_o}\right]^{-1}\dot{E}_i \cos\theta. \tag{70}$$

Eq. (70) contains the term of cos θ, so the output of demodulating circuit varies with the phase of carrier wave. Combining Eq. (70) with Eq. (59) or Eq. (63), we obtain the whole transfer function which is

$$Y(s) = 2Y'_{pc}(s)\frac{\Delta'}{\Delta}(s)\cdot\frac{a_1\cos\theta}{B\infty\left[(2-a_o)\frac{A\infty}{B\infty}+a_o\frac{\dot{A}_o}{\dot{B}_o}\right]}, \tag{71}$$

in casse where a parallel resonant element is contained in modulating circuit, and which is

$$Y(s) = 2\omega_c L\ Y_{sc}'(s)\ \frac{\Delta'}{\Delta}(s) \cdot \frac{a_1 \cos\left(\theta + \frac{\pi}{2}\right)}{B\infty\left[(2-a_0)\frac{A\infty}{B\infty} + a_0\frac{\dot{A}_0}{\dot{B}_0}\right]} \tag{72}$$

in case where a series resonant element is contained.

Ⅺ. CONCLUSION

In this paper, the transfer function of chopper circuit consisting of only C and R is derived theoretically, and the author shows that the transfer function of chopper circuit containing a resonant element can be calculated, if Q of the resonant element is sufficiently high.

(Received September 26, 1959)

Acknowledgement

The author thanks Mr. Tatsuo Ôtake who has earnestly pefrormed the experiment of the transient phenomena, and his colleagues, especially Mr. Tsunetomo Annô Professor, of the Yamagata University, for their kind helps.

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NAKATSUYAMA : An Analysis of Chopper Circuits

チョッパ回路の一解析

中津山幹男

工学部電気工学科

チョッパ回路の伝達関数については，既に，二，三の論文が発表されているが，チョッパ回路の構成が少し複雑になると，伝達関数の計算がやゝ困難になってくる。本論文では，まず，CR のみよりなる回路に考察を限定して，チョッパ端子の電圧，電流の性質に着目し，この性質からチョッパ回路の伝達関数を求める公式を得た。この公式は，チョッパ回路の F－マトリックス素子から，直ちに，求められるので，計算が簡単で甚だ便利であり，チョッパ変調回路と，復調回路の各々のチョッパの接触率や，位相が異なっている場合にも適用できる。回路素子が与えられたとき，利得を最大ならしめるチョッパの接触率の値もこの公式から誘導した。

回路に，L が含まれている場合は，適当な条件が満足されれば，CR 回路と同様にその伝達関数を誘導することができる。更に，チョッパ変調回路が，LC 共振回路を含む場合でも，そのQが高ければ，変調回路の出力は，平衡変調波となり，その包路線の伝達関数の公式を求めることができた。平衡変調波の復調出力も容易に得られる。